

# CONFORMAL EXTENSIONS OF FUNCTIONS DEFINED ON ARBITRARY SUBSETS OF RIEMANN SURFACES

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**ABSTRACT.** For a function defined on an arbitrary subset of a Riemann surface, we give conditions which allow the function to be extended conformally. One folkloric consequence is that two common definitions of an analytic arc in  $\mathbb{C}$  are equivalent.

The purpose of this note is to extend conformally a function given on an arbitrary subset of a Riemann surface. Our original motivation was to prove that a construction of Nestoridis and Zadik in [3] holds more generally, so there is no need to use the specific form of the extended function used in [3, Prop. 2.3, iii]. A corollary of our result is that two common definitions of analytic arcs in  $\mathbb{C}$  are equivalent.

**Definition 1.** *A (topological or Jordan) open arc  $J$  in a topological space  $X$  is a subset  $J \subset X$ , which is a homeomorphic image of the open unit interval*

$$I = \{t \in \mathbb{R} : 0 < t < 1\},$$

*equivalently, of the real line. Thus, if  $J$  is an open arc, there is a parametrization  $\varphi : I \rightarrow J$ , which is a homeomorphism.*

If we are considering an open arc in a Riemann surface  $X$ , then topological notions (closure, boundary, etc.) will be with respect to  $X$ . In particular, if we are considering an open arc in the Riemann sphere  $\overline{\mathbb{C}}$ , then topological notions will be with respect to  $\overline{\mathbb{C}}$ . For a holomorphic mapping  $f : X \rightarrow Y$  between two Riemann surfaces and a point  $p \in X$ , we write  $f'(p) = 0$  to signify that, for some (hence any) local coordinate mappings  $\varphi$  and  $\psi$  at  $p$  and  $f(p)$  respectively, with  $\varphi(p) = 0$ , we have  $(\psi \circ f \circ \varphi^{-1})'(0) = 0$ . In particular, either  $X$  or  $Y$  may be the Riemann sphere  $\overline{\mathbb{C}}$ .

For an open arc  $J$  in  $X$  with parametrization  $\varphi$ , we define the initial end  $J(0)$  and the terminal end  $J(1)$  of  $J$  as

$$J(0) = \bigcap_{0 < t < 1} \overline{\varphi(0, t]}, \quad J(1) = \bigcap_{0 < t < 1} \overline{\varphi[t, 1]}.$$

Since  $\varphi$  is a homeomorphism onto  $J$  and  $J$  has the relative topology induced by  $X$ , it follows that the open arc  $J$  is disjoint from both of its ends. Each end is a closed connected set. If the initial end is a point, we call this the initial point of the arc (even though it is not on the arc). Similarly, if the terminal end is a point, we call it the terminal point.

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If  $\varphi : I \rightarrow J$  is an open arc and  $f : J \rightarrow \overline{\mathbb{C}}$ , we define the initial and terminal cluster sets of  $f$  on  $J$ :

$$C_0(f, J) = \bigcap_{t \in I} \overline{\{f(\varphi(s)) : 0 < s < t\}}, \quad C_1(f, J) = \bigcap_{t \in I} \overline{\{f(\varphi(s)) : t < s < 1\}}.$$

The cluster set  $C(f, J)$  is the union of the initial cluster set of  $f$  and the terminal cluster set of  $f$ . In other words,

$$C(f, J) = C_0(f, J) \cup C_1(f, J) = \bigcap_{\epsilon > 0} \overline{(f \circ \varphi)(I \setminus [\epsilon, 1 - \epsilon])}.$$

**Theorem 1.** *Suppose  $J$  is an open arc in  $\overline{\mathbb{C}}$  and  $f : U \rightarrow \overline{\mathbb{C}}$  is a holomorphic mapping on an open neighborhood  $U$  of  $J$  in  $\overline{\mathbb{C}}$ . Suppose  $f|_J$  is injective,  $f'(z) \neq 0$ , for  $z \in J$ , and the sets  $f(J)$  and  $C(f, J)$  are disjoint. Then,  $f$  is injective (one-to-one conformal) on some neighborhood of  $J$ .*

*Proof.* Assume, first of all, that the initial and terminal cluster sets  $C_0(f, J)$  and  $C_1(f, J)$  are disjoint. Consider the mapping  $\psi : I \rightarrow f(J)$ , given as  $\psi = f \circ \varphi$ . As a composition of continuous injective mappings,  $\psi$  is also continuous and injective. We claim that  $\psi^{-1}$ , (which is well-defined) is also continuous. Suppose for the sake of contradiction, that there is a sequence  $\psi(t_j), t_j \in I$ , which converges to a point  $\psi(\alpha), \alpha \in I$ , but  $t_j \not\rightarrow \alpha$ . We may assume that  $t_j$  converges to point  $\beta \in [0, 1]$ . If  $\beta \in I$ , then  $\psi(t_j) \rightarrow \psi(\beta) \neq \psi(\alpha)$ , which is a contradiction. If  $\beta = 0$ , then  $t_j \rightarrow 0$ , so  $\psi(\alpha) = \lim \psi(t_j) \in C_0(f, J)$ , since  $t_j \rightarrow 0$ , which again is a contradiction, since  $f(J)$  is disjoint from  $C_0(f, J)$ . The same argument shows that  $t_j$  cannot converge to 1. Thus,  $\psi^{-1}$  is continuous and so  $\psi$  is a homeomorphism. This shows that  $f(J)$  is also an open arc.

Let  $W_o$  be the component of  $\overline{\mathbb{C}} \setminus C_o(f, J)$  which contains the connected set  $f(J) \cup C_1(f, J)$  and let  $W_1$  be the component of  $\overline{\mathbb{C}} \setminus C_1(f, J)$  which contains the connected set  $f(J) \cup C_o(f, J)$ . Let  $\widehat{C}_o(f, J)$  be the union of  $C_o(f, J)$  with all of its complementary components in  $\overline{\mathbb{C}}$  which do not meet the connected set  $f(J) \cup C_1(f, J)$ . We define  $\widehat{C}_1(f, J)$  similarly. We may also say that  $\widehat{C}_o(f, J) = \overline{\mathbb{C}} \setminus W_o$  and  $\widehat{C}_1(f, J) = \overline{\mathbb{C}} \setminus W_1$ .

The sets  $f(J)$ ,  $\widehat{C}_o(f, J)$  and  $\widehat{C}_1(f, J)$  are pairwise disjoint. Both compact connected sets  $\widehat{C}_o(f, J)$  and  $\widehat{C}_1(f, J)$  have only one complementary component  $W_o$  and  $W_1$  respectively in  $\overline{\mathbb{C}}$  which both contain  $f(J)$ . There exists a homeomorphism

$$h : \overline{\mathbb{C}} \setminus [\widehat{C}_o(f, J) \cup \widehat{C}_1(f, J)] \rightarrow \overline{\mathbb{C}} \setminus \{p_o, p_1\},$$

mapping the topological annulus  $\overline{\mathbb{C}} \setminus [\widehat{C}_o(f, J) \cup \widehat{C}_1(f, J)]$  onto the twice punctured sphere  $\overline{\mathbb{C}} \setminus \{p_o, p_1\}$ , where  $p_o$  and  $p_1$  are distinct finite points. The homeomorphism  $h$  maps the open arc  $f(J)$  to the open arc  $h(f(J))$ , whose initial and terminal points are respectively  $p_o$  and  $p_1$ . By composing with a Möbius transformation, we may assume that the open arc  $h(f(J))$  does not pass through  $\infty$ . Let  $\alpha_o$  and  $\alpha_1$  be disjoint open arcs in  $\mathbb{C}$ , where  $\alpha_o$  joins  $\infty$  to  $p_o$  and  $\alpha_1$  joins  $p_1$  to infinity. Set

$$\alpha = \alpha_o \cup \{p_o\} \cup h(f(J)) \cup \{p_1\} \cup \alpha_1.$$

The open arc  $h(f(J))$  is the homeomorphic image of the open unit interval  $I$  under the parametrization  $h \circ f \circ \varphi$  with initial point  $p_o$  and terminal point  $p_1$ . We may extend this to a parametrization  $\eta : (-\infty, +\infty) \rightarrow \alpha$  of the open arc  $\alpha$ . By the

Schoenflies Theorem [2, page 81], we may further extend  $\eta$  to a homeomorphism  $\eta : \mathbb{C} \rightarrow \mathbb{C}$ . Let us denote the homeomorphism

$$h^{-1} \circ \eta : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{C} \setminus [\widehat{C}_o(f, J) \cup \widehat{C}_1(f, J)]$$

by  $H$ .

Fix  $t \in I$ . We may choose a closed disc  $\overline{D}_t$  with center  $t$  so small that  $0, 1 \notin \overline{D}_t$  and so  $H$  maps  $D_1$  homeomorphically onto a Jordan domain  $W_t$  containing  $f(\varphi(t))$ . Since  $f'(\varphi(t)) \neq 0$ , we may choose a branch  $g_t$  of  $f^{-1}$  in a neighborhood of  $f(\varphi(t))$ , such that  $f^{-1} \circ f$  is the identity in a neighborhood of  $\varphi(t)$ . We may choose  $D_t$  so small that the Jordan domain  $H(D_t)$  is contained in the domain of definition of this inverse branch  $g_t$ . We claim that these inverse branches, for various  $t$  are compatible. Indeed, since  $H$  is a homeomorphism, two Jordan domains  $H(D_s)$  and  $H(D_t)$  intersect if and only if the discs  $D_s$  and  $D_t$  intersect and in this case the intersection  $H(D_s) \cap H(D_t)$  has only one component, which is  $H(D_s \cap D_t)$ . Since  $f$  is injective on  $J$ , the branches  $g_s$  and  $g_t$  agree on the (non-empty) arc of  $f(J)$  in  $H(D_s) \cap H(D_t)$ . By the uniqueness principle,  $g_s = g_t$  on  $H(D_s) \cap H(D_t)$ . We have verified that the inverse branches  $g_t, t \in I$ , are compatible. Thus, we may define a branch  $g$  of  $f^{-1}$  on the neighborhood  $W = \cup_{t \in I} H(D_t)$  of  $f(J)$ . We have that  $f$  maps the open neighborhood  $g(W)$  of  $J$  biholomorphically onto the neighborhood  $W$  of  $f(J)$ . This completes the proof, in case the initial and terminal cluster sets  $C_o(f, J)$  and  $C_1(f, J)$  are disjoint.

Suppose the initial and terminal cluster sets  $C_o(f, J)$  and  $C_1(f, J)$  are not disjoint. The cluster set  $C(f, J)$  is then a continuum or a point, so the open set  $\overline{\mathbb{C}} \setminus C(f, J)$  is simply connected. In particular, the component  $\Omega$  of  $\overline{\mathbb{C}} \setminus C(f, J)$  which contains the connected set  $f(J)$  is simply connected. Set  $E = \overline{\mathbb{C}} \setminus \Omega$ . If  $E$  is a singleton, we may map  $\Omega$  to  $\mathbb{C}$  by a Möbius transformation so that  $E$  goes to  $\infty$ . Suppose  $E$  is a continuum. By the Riemann mapping theorem, we may assume that  $\Omega$  is the unit disc and  $f(J)$  is an open arc in  $\Omega$ . There is a homeomorphism  $h : \Omega \rightarrow \mathbb{C}$  so that  $h(w) \rightarrow \infty$ , as  $|w| \nearrow 1$ . Thus, whether  $E$  is a singleton or not, there is a homeomorphism from  $\Omega$  to  $\mathbb{C}$ . After this mapping, the image of  $f(J)$  is an open arc both ends of which are  $\infty$ . We may parametrize  $f(J)$  by the real line and use the Schoenflies theorem as above.  $\square$

This result extends to arcs on Riemann surfaces. For that purpose, the following lemma is useful and of independent interest.

**Lemma 1.** *Let  $J$  be an open arc in a Riemann surface  $X$ . Then,  $J$  has a fundamental system of simply connected neighborhoods.*

*Proof.* For simplicity of notation, it will be convenient to consider  $J$  as being parametrized by  $\mathbb{R}$  rather than  $(0, 1)$ . Thus,  $\varphi : (-\infty, +\infty) \rightarrow J$ . We may choose an increasing sequence  $t_j, j \in \mathbb{Z}$ , such that  $\lim_{j \rightarrow \pm\infty} t_j = \pm\infty$  and each  $\varphi(t_j, t_{j+1}) = J_j$  is contained in a chart  $U_j$ . We may construct Jordan domains  $V_j$ , such that  $J_j \subset V_j \subset U_j$ , the end points of  $J_j$  are on  $\partial V_j$  and the  $\overline{V}_j$  are disjoint except possibly for end points. By construction, the set  $V = \cup_j V_j$  covers  $J \setminus \cup_j \{\varphi(t_j)\}$ . By introducing suitable small neighborhoods of the end points  $\varphi(t_j)$ , we may enlarge  $V$  to a “strip”  $S$  which is homeomorphic to  $\{z = x + iy : -\infty < x < +\infty, |y| < 1\}$  and hence simply connected. If  $W$  is a neighborhood of  $J$ , we can replace  $X$  by the component of  $W$  containing  $J$ . We have thus shown that  $J$  has a fundamental system of simply connected neighborhoods.  $\square$

**Theorem 2.** *Let  $J$  be an open arc in a Riemann surface  $X$  and  $f : U \rightarrow \overline{\mathbb{C}}$  a holomorphic mapping on an open neighborhood  $U$  of  $J$  in  $X$ . Suppose  $f|_J$  is injective,  $f'(p) \neq 0$ , for  $p \in J$ , and the sets  $f(J)$  and  $C(f, J)$  are disjoint. Then,  $f$  is injective (one-to-one conformal) on some neighborhood of  $J$ .*

*Proof.* We may assume that  $U$  is simply connected and so it is conformally equivalent to  $\overline{\mathbb{C}}$ , to  $\mathbb{C}$  or to the unit disc. The case  $\overline{\mathbb{C}}$  is excluded, since  $f$  is not constant. The conclusion now follows from the previous theorem.  $\square$

We shall now extend our results on arcs to arbitrary sets. Of course, a necessary condition that a function be extendable conformally is that it be extendable holomorphically. To this end we have the following.

**Theorem 3.** *Let  $X$  be a Riemann surface,  $E$  an arbitrary subset of  $X$  and  $f : E \rightarrow Y$  a mapping from  $E$  to a complex manifold  $Y$ , such that, for each  $p \in E$ , there is an open neighborhood  $U_p \subset X$  of  $p$  and a holomorphic mapping  $\Phi_p : U_p \rightarrow Y$ , such that  $\Phi_p(q) = f(q)$ , for all  $q \in U_p \cap E$ . Then,  $f$  extends to a holomorphic mapping  $\Phi : U \rightarrow Y$  on some neighborhood  $U$  of  $E$ , such that  $\Phi$  locally coincides with some  $\Phi_p$ .*

A form of this theorem was proved in [1, Th. 5] for the special case of meromorphic functions, that is, when  $Y = \overline{\mathbb{C}}$ . The theorem in [1] is also weaker in the sense that it is not claimed that,  $\Phi$  locally coincides with some  $\Phi_p$  but merely that  $\Phi(p) = f(p)$ .

*Proof.* The proof is a modification of an argument taken from [1]. Let  $E'$  be the set of accumulation points of  $E$  which are in  $E$ . Choose a distance function on  $X$  (see [6]). For  $p \in E$ , denote by  $r_p$  the distance of  $p$  from  $\partial U_p$ . For each  $p \in E'$ , we choose a parametric disc  $D_p$  for  $X$  at  $p$ , such that  $\text{diam} D_p < r_p/2$ .

Claim: for every two such discs  $D_p, D_q$ , with  $p, q \in E'$ , we have

$$\Phi_p(z) = \Phi_q(z), \quad \text{for all } z \in D_p \cap D_q.$$

We may suppose that  $D_p \cap D_q \neq \emptyset$  and  $\text{diam} D_q \leq \text{diam} D_p$ . Then,  $D_q \subset U_p$ . Since  $q$  is a limit point of  $E$ , and both  $\Phi_p$  and  $\Phi_q$  equal  $f$  on  $E \cap D_q$ , it follows that  $\Phi_p = \Phi_q$  on the component of  $U_p \cap U_q$  containing  $D_q$ . Obviously, this component contains  $D_p \cap D_q$  so the claim follows.

We have

$$E' \subset U' \stackrel{\text{def}}{=} \bigcup_{p \in E'} D_p.$$

By the claim, we may define a holomorphic function  $\Phi$  on the open neighborhood  $U'$  of  $E'$ , by setting  $\Phi = \Phi_p$  on each  $D_p, p \in E'$ . Moreover,  $\Phi = f$  on  $U' \cap E$ .

Arrange the points of  $E \setminus U'$  in a sequence  $p_n$ . Denote by  $U_p$  the neighborhood of  $p$ . For each  $p = p_n$ , choose a disc  $D_n = D_p$  centered at  $p$  and contained in  $U_p$  so small that the radius is less than  $1/n$ , and such that  $E \cap \overline{D_p} = \{p\}$ . We can also arrange that the discs  $D_n$  are pairwise disjoint. For instance, it suffices that the radius of  $D_n$  be smaller than  $1/2$  the distance of  $p_n$  to the rest of  $E$ . Set  $\Phi = \Phi_{p_n}$  on  $D_n$ . Let  $U$  be defined as

$$U = [U' \setminus \bigcup_{p \in E \setminus U'} \overline{D_p}] \cup \bigcup_{p \in E \setminus U'} D_p.$$

In order to check that the set  $U$  is open one can use the fact that the radii of the  $D_n$  converge to zero. The mapping  $\Phi$  is well defined on the neighborhood  $U$  of  $E$  and has the desired properties.  $\square$

**Remark 1.** In the proof of Theorem 3 the function  $\Phi$  locally coincides with some  $\Phi_p$ . Therefore, if we assume that the derivative of  $\Phi_p$  at  $p$  is non zero, then we can consider smaller open sets  $U_p$  so that the derivative of  $\Phi_p$  is everywhere non zero. It follows that the derivative of  $\Phi$  is non zero everywhere on  $U$  and for every  $z$  in  $U$  the mapping  $\Phi$  is locally a homeomorphism between two open sets containing  $z$  and  $\Phi(z)$ , respectively.

A **holomorphic curve** in a complex manifold  $Y$  is a nonconstant holomorphic mapping  $\Phi : X \rightarrow Y$  from a Riemann surface  $X$  into  $Y$ .

**Corollary 1.** *Let  $X$  be a Riemann surface,  $E$  a connected subset of  $X$  and  $f : E \rightarrow Y$  a mapping from  $E$  to a complex manifold  $Y$ , such that, for each  $p \in E$ , there is an open neighborhood  $U_p \subset X$  of  $p$  and a holomorphic mapping  $\Phi_p : U_p \rightarrow Y$ , such that  $\Phi_p(q) = f(q)$ , for all  $q \in U_p \cap E$ . Then,  $f$  extends to a holomorphic curve  $\Phi : V \rightarrow Y$  mapping some connected neighborhood  $V$  of  $E$  into  $Y$ , such that  $\Phi$  locally coincides with some  $\Phi_p$ .*

*Proof.* In Theorem 3, let  $V$  be the component of  $U$  containing  $E$ . Then,  $V$  is a Riemann surface and so  $\Phi : V \rightarrow Y$  is, by definition, a holomorphic curve.  $\square$

The Corollary applies, in particular, to the case that  $E$  is an open arc.

In order to extend a function, not only holomorphically, but even biholomorphically, the following lemma [4, Lemma 3.6] is helpful.

**Lemma 2.** *Let  $U, Y$  be Hausdorff spaces with countable bases and  $U$  be locally compact. If  $\Phi : U \rightarrow Y$  is a local homeomorphism and the restriction of  $\Phi$  to a closed subset  $E$  is a homeomorphism, then  $\Phi$  is a homeomorphism on some neighbourhood  $V$  of  $E$ .*

Let  $E$  be a subset of a Riemann surface  $X$ , and let  $f : E \rightarrow Y$ . For a point  $p \in X$ , we define the cluster set  $C(f, p)$  of  $f$  at  $p$  as the set of all values  $w \in Y$ , such that there is a sequence  $z_n \in E, z_n \rightarrow p$ , for which  $f(z_n) \rightarrow w$ .

**Theorem 4.** *Let  $X$  and  $Y$  be Riemann surfaces and  $E$  be an arbitrary subset of  $X$ . Suppose, for a function  $f : E \rightarrow Y$ , that the cluster sets  $C(f, p), p \in X$ , are pairwise disjoint and, for each  $p \in E$ , there is an open neighborhood  $U_p \subset X$  of  $p$  and a holomorphic mapping  $\Phi_p : U_p \rightarrow Y$ , such that  $\Phi_p(q) = f(q)$ , for all  $q \in U_p \cap E$  and  $\Phi'_p(p) \neq 0$ . Then  $f$  extends to a one-to-one conformal mapping of some open neighborhood  $V$  of  $E$  onto an open subset of  $Y$ .*

*Proof.* By Theorem 3,  $f$  extends to a holomorphic mapping into  $Y$ . According to Remark 1, for every  $z$  in  $U$ , the mapping  $\Phi$  is locally a homeomorphism between two open sets containing  $z$  and  $\Phi(z)$ , respectively. Set  $F = f(E)$ . We claim that  $f : E \rightarrow F$  is a homeomorphism. First of all,  $f$  is continuous, because it is the restriction of the continuous function  $\Phi$ . The continuity of  $f$  implies that  $C(f, z) = f(z)$ , for all  $z \in E$ . The hypothesis on cluster sets therefore implies that  $f$  is injective and hence has an inverse function  $\psi : F \rightarrow E$ . We claim that  $\psi$  is continuous. To see this, let  $b = f(a)$  be a point of  $F$  and suppose  $w_n = f(z_n)$  converges to  $b$ . Let  $a'$  be any limit point of  $\psi(w_n) = z_n$ . Then, from the definition of  $C(f, a')$ , it follows that  $b \in C(f, a')$ . Since  $b$  is also in  $C(f, a)$ , the hypothesis on cluster sets implies

that  $a' = a$ . We have shown that  $\psi(w_n) \rightarrow a = \psi(b)$ . This confirms the claim that  $\psi$  is continuous and also that  $f$  is a homeomorphism.

If  $E$  is closed, the theorem now follows from Lemma 2.

In general, fix some distance function on  $Y$ . For each  $z \in E$ , we may choose an open neighborhood  $\tilde{U}_z$  such that  $\Phi(\tilde{U}_z)$  is a disc  $D_w$  centered at  $w = f(z)$  and we may choose a branch  $\Psi_w$  of  $\Phi^{-1}$  in  $D_w$  such that  $\Psi_w \circ \Phi$  is the identity on  $\tilde{U}_z$ . We claim that  $\Psi_w = \psi$  on  $F \cap D_w$ . To verify the last claim, suppose  $b \in F \cap D_w$ . From the definition of  $\Psi_w$ , there is a point  $a \in E \cap \tilde{U}_z$ , such that  $f(a) = b$  and  $a = \Psi_w(b)$ . Since  $f$  is injective,  $a = f^{-1}(b) = \psi(b)$ . We have shown that  $\Psi_w(b) = a = \psi(b)$ , which establishes the claim. By Theorem 3, there is an open neighborhood  $W$  of  $F$  and a holomorphic mapping  $\Psi : W \rightarrow X$ , such that  $\Psi = \Psi_w$  on  $D_w$ , for each  $w \in F$ . The mapping  $\Psi$  is holomorphic on  $W$  and maps  $W$  to an open neighborhood  $V$  of  $E$ , contained in the domain of definition of  $\Phi$ . Moreover,  $\Psi \circ \Phi$  is the identity on  $V$ . Thus,  $\Phi$  is biholomorphic from  $V$  to  $W$  and  $\Phi$  restricted to  $E$  is  $f$ . This finishes the proof.  $\square$

We remark that if  $E$  has no isolated points, then the extension in Theorem 4 is unique in the sense that any two extensions agree on some neighborhood of  $E$ . In particular, this applies to the case that  $E$  is a curve.

**Definition 2.** *[analytic arc] An analytic open arc  $J$  in a Riemann surface  $X$  is an arc in  $X$ , whose parametrization  $\varphi$  is analytic. Thus, for every  $t_0 \in I = (0, 1)$  and local coordinate  $z$  in a neighborhood of  $\varphi(t_0)$ , the function  $z \circ \varphi$  has a representation as a power series near  $t_0$  :  $(z \circ \varphi)(t) = \sum a_j(t - t_0)^j$ . We shall say that  $J$  is a regular open analytic arc if  $\varphi'(t) \neq 0$ , for all  $t \in (0, 1)$ .*

**Definition 3.** *[conformal arc] A conformal open arc  $J$  in a Riemann surface is an arc in  $X$ , with a parametrization which extends to a one-to-one conformal mapping from a neighborhood of  $I \subset \mathbb{C}$  into  $X$ .*

**Corollary 2.** *An open arc  $J$  in a Riemann surface  $X$  is a regular analytic arc if and only if it is a conformal arc.*

*Proof.* Clearly, if  $J$  is a conformal arc, then it is a regular analytic arc.

Suppose, conversely, that  $J$  is a regular analytic arc. Consider firstly the case that  $X = \mathbb{C}$ . Thus, there is a parametrization  $\varphi : I \rightarrow J$  which is analytic and such that  $\varphi'(t) \neq 0$ , for  $t \in I$ . For each  $t \in I$ , we may extend  $\varphi$  to a holomorphic function  $\varphi_t$  in a disc  $D_t$  centered at  $t$  and disjoint from  $\{0, 1\}$ . If  $D_s \cap D_t \neq \emptyset$ , then  $\varphi_s = \varphi = \varphi_t$  on  $I \cap D_s \cap D_t$ . Thus,  $\varphi_s = \varphi_t$  on  $D_s \cap D_t$ . By setting  $\varphi = \varphi_t$  on  $D_t$ , for each  $t \in I$ , we obtain a well-defined holomorphic extension of  $\varphi$  to the open neighborhood  $U = \cup_{t \in I} D_t$  of  $I$ .

We note that the initial and terminal ends of  $J$  are the same as the initial and terminal cluster sets  $C_o(\varphi, I)$  and  $C_1(\varphi, I)$ . Since  $\varphi : I \rightarrow J$  is a homeomorphism, these cluster sets are disjoint from  $J$ . It follows directly from Theorem 4 that  $J$  is a conformal arc. This completes the proof of the converse, in case  $X = \mathbb{C}$ .

In the general case, it follows from Lemma 1, that there is a simply connected neighborhood  $U$  of  $J$  in  $X$ . By the Riemann mapping theorem,  $U$  is conformally equivalent to an open subset of  $\mathbb{C}$ , and so it follows from the case just treated that  $J$  is a conformal arc.  $\square$

This equivalence, stated in the corollary, is probably folkloric, at least in case the analytic arc is the image of the *closed* rather than open unit interval or in case

we have a Jordan curve. Osserman [5] says that an analytic Jordan curve on a Riemann surface  $X$  is defined by an analytic mapping of the unit circle into  $X$  and this mapping extends to a conformal mapping of an annulus into  $X$ . In these cases, one can use compactness to give a simpler proof than the one we have given.

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